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Irreducible Representations of Witten's Deformations of $U(sl_2)$

Rajesh S. Kulkarni

E-mail: rkulkarn@iu-math.math.indiana.edu

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Using E. Witten's approach based on duality in conformal field theory, one can obtain a 7-parameter family of deformations of $U(sl_2)$ (called the "Witten's Deformations"). Le Bruyn singled out a subfamily (called "conformal sl_2 algebras") and studied their finite dimensional representations. Here we consider all of Witten's deformations. We compute the left spectrum of these algebras and thereby classify irreducible representations for generic values of parameters and finite dimensional representations in certain "roots of unity" cases. © 1999 Academic Press

1. INTRODUCTION

In using the "vertex models" to give an explanation of existence of quantum groups, Witten introduced a 7-parameter family of deformations of $U(sl_2)$. These are associative algebras generated by x, y, z and are described by relations

$$
\begin{aligned}
xz - \alpha zx &= \beta x \\
zy - \epsilon yz &= \zeta y \qquad (1.1)\\
yx - \gamma xy &= \delta z^2 + \sigma z,
\end{aligned}
$$

where α, β, ϵ, ζ, γ, δ, and σ are in the field $\mathbf{C}$. The parameters depend on the coupling constant of a specific physical theory. For example, see [W2] for the evaluation of these parameters in case of three dimensional Chern–Simons gauge theory. The interested reader may refer to [W1, W2] for background information.

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In his article [L], Le Bruyn gives an exposition of some of the essential part of [W2], especially the derivation of the deformations mentioned above. These algebras have a natural filtration. However, generically, their associated graded algebras are not commutative. Le Bruyn singled out a 3-parameter family: those deformations whose associated graded algebras are Auslander regular (called conformal sl_2 algebras). Their defining relations may be given as

$$\begin{aligned} zx - axz &= x \\ yz - azy &= y \\ xy - cyx &= bz^2 + z, \end{aligned} \tag{1.2}$$

where $a, b, c \in \mathbf{C}$. Finite dimensional representations of these algebras are studied in [L]. There the approach is based on recently developed tools from noncommutative projective geometry. (See, for example, [ATV1, ATV2]. Also this approach is compared with the more traditional approach of studying some Borel-like algebras.

In this paper, another approach is used to study these algebras. It uses some basic tools from noncommutative algebraic geometry which have been developed by A. L. Rosenberg (see [R1]). This theory uses the left spectrum of a noncommutative ring as defined in [R1]. From the point of view of representation theory an important feature of it is that infinite as well as finite dimensional representations are studied.

Using this theory, we classify all the irreducible representations of these algebras for generic values of parameters. In the remaining roots of unity cases, the methods to classify the representations are indicated. For some of these cases, we make explicit calculations exhibiting these methods. (See Section 4.2).

It is shown that (see Section 3) the algebras given by (1.1) can be classified into three families:

(1) Conformal sl_2 algebras given by relations (1.2).

(2) $U_2(k, \zeta)$ whose relations are given by

$$xz - zx = x, \qquad zy - yz = \zeta y, \qquad yx - \gamma xy = 0.$$

(3) The quantum plane whose relations are given by

$$wx = \alpha xw, \qquad yw = \epsilon wy, \qquad xy = \gamma yx.$$

It turns out that the conformal sl_2 algebras are hyperbolic rings. These are particularly convenient for the study of representations. They are viewed as skew polynomial rings over commutative rings with commuting relations given by some automorphism θ of the coefficient ring. The problem of computing the left spectrum is then reduced to the study of left ideals lying over the prime ideals in the coefficient ring. Then we have theorems [R1] describing the left spectrum depending upon the lengths of the orbits of prime ideals under the action of θ.

The other cases referred to above are not hyperbolic rings. But they can be thought of as skew polynomial rings over skew polynomial rings. First we compute the left spectrum of the base skew polynomial ring either by direct methods or by treating it as a hyperbolic ring. Then we may use the theory of the spectrum of hyperbolic categories to get representations in these cases.

Now we briefly describe the contents of the article.

In the next section, which is essentially background for the paper, some preliminaries are sketched. The left spectrum of any associative ring is defined, followed by the definition of hyperbolic rings. Some theorems which we will use in the following sections are stated without proofs. The proofs of these theorems and other related theorems can be found in [R1].

In Section 3, we prove a theorem about isomorphism classes of these algebras. This classification is convenient for the theory which we will use to compute the left spectrum (and hence representations).

In Section 4, we describe the left spectrum and the irreducible representations of the conformal sl_2 algebra $U_{abc}(sl_2)$. We give all the computations of the left spectrum for the non-degenerate case $a \neq 1$. First we deal with the case of infinite orbits. Then the generic case of those left ideals whose intersection with the coefficient ring is zero is dealt with in the following subsection. The case of finite orbits remains which we do next. We show the reduction of this case to the θ-invariant left ideals and give a complete list of representations. Finite dimensional representations (in case of algebraically closed field of characteristic zero) are computed in a separate subsection. For the degenerate case, we give some formulas to observe that the analysis follows the earlier pattern. Of course this degenerate case is important as the enveloping algebra, which in our notation is $U_{1,0,1}(sl_2)$ and belongs to this case.

In Section 5, we consider two sporadic cases which are described in Theorem 3.0.3. The category of their representations is a hyperbolic category. This is shown by proving that these algebras are double skew polynomial extensions. To compute representations, we assume that the field k is algebraically closed and is of characteristic zero. First, we compute the left spectrum of the base skew polynomial rings. Then we indicate how to obtain representations of the extension of these skew polynomial algebras with partial results.

Remark. Unless otherwise mentioned, we consider algebras given by relations in (1.1) and (1.2) with coefficients in arbitrary fields.

2. PRELIMINARIES

For notions introduced in this section and proofs of propositions stated, the general reference is [R1].

We introduce the notion of the left spectrum of rings and abelian categories. We will always be interested in the abelian category *R-mod* of left modules over a ring R.

Let $\succ$ be the following relation (preorder) on the set $I_l R$ of left ideals of an associative ring R: $m \succ n$ if there exists a finite subset w of R such that the left ideal $(m : w) = \{r \in R | rw \subset m\}$ is contained in n.

DEFINITION 2.0.1. The left spectrum $Spec_l R$, of the ring R consists of all the left ideals p of R such that $(p : x) \succ p$ for each $x \in R - p$.

The preorder $\succ$ mentioned above restricts to $Spec_l R$ and is in fact an equivalence relation on this set. Denote by $\mathbf{Spec}_l R$ the equivalence classes of $Spec_l R$ under this relation. The left spectrum, $Spec_l R$, contains the set $\mathrm{Max}_l R$ of maximal left ideals of R. Under localization by any Serre subcategory $\mathbf{S}$, the left spectrum of the ring R is mapped to the left spectrum of $\mathbf{S}^{-1}R$.

The definition of left spectrum above can be generalized to the spectrum of an abelian category as follows. Let $\mathscr{A}$ be an abelian category (here, the category *R-mod* of left R-modules); and let M, N be objects of $\mathscr{A}$. Write $M \succ N$ if there exists a diagram

$$(l)M \leftarrow L \rightarrow N,$$

where $(l)M$ is the direct sum of l copies of M; the first arrow is a monomorphism and the second arrow is an epimorphism.

DEFINITION 2.0.2. The spectrum $Spec\mathscr{A}$, is the collection of all the objects M of $\mathscr{A}$ such that $N \succ M$ for any nonzero subobject N of M.

As in the case of *R-mod*, the preorder $\succ$ mentioned above restricts to $Spec\mathscr{A}$ and is an equivalence relation. Denote by $\mathbf{Spec}\mathscr{A}$ the equivalence classes of $Spec\mathscr{A}$ under this relation.

We will need mostly the following fact about these spectra:

PROPOSITION 2.0.1. *Let $\mathscr{A}$ be the category R-mod of left modules over a ring R. Then the map $Spec_l R \to Ob\mathscr{A}$ assigning to a left ideal p the quotient module R/p induces a bijection of the sets of equivalence classes* $\mathbf{Spec}_l R \to \mathbf{Spec}\mathscr{A}$.

The following definitions will be used in the main text:

DEFINITION 2.0.3. Let $\langle P \rangle \in \mathbf{Spec}\mathscr{A}$ denote the equivalence class of an object $P \in Spec\mathscr{A}$.

(1) The support of an object M in an abelian category $\mathscr{A}$ is the collection $\mathrm{Supp}(M)$ of all $\langle P\rangle \in \mathbf{Spec}\mathscr{A}$ such that $M \succ P$.

(2) For any object M of $\mathscr{A}$, $\mathrm{Ass}(M)$ is the collection of $\langle P\rangle \in \mathbf{Spec}\mathscr{A}$ for each P which is a subobject of M. The points of $\mathrm{Ass}(M)$ are called associated points to M.

If R is a commutative ring, then $Spec_l R$ is same as the prime spectrum of ring R. (The prime spectrum of a ring R consists of prime ideals of R and is denoted by $Spec R$.) The most important notion for our purposes is the notion of a hyperbolic ring which we introduce after the following definitions:

DEFINITION 2.0.4. Let $R \hookrightarrow S$ be a ring monomorphism. A left ideal $\mathbf{p}$ of the ring S is said to lie over a left ideal p of R if $\mathbf{p} \cap R = p$.

Let θ denote an automorphism of a commutative ring R and let ξ be an element of R. Denote by $R\{\theta, \xi\}$ the algebra generated by the indeterminates x, y with relations

$$xa = \theta(a)x, \qquad ya = \theta^{-1}(a)y \text{ for any } a \in R,$$

and

$$xy = \xi, \qquad yx = \theta^{-1}(\xi).$$

It turns out that these rings are especially convenient for studying their left spectrum and hence the representations (cf. [R1]). The left spectrum is classified by left ideals lying over prime ideals of R. The classification depends on length of the orbit of p under the action of the automorphism θ. The spectrum of R is divided into four cases:

(a) $p = \{0\}$.

(b) Prime ideals p such that $\theta^n(p) \neq p$ for any $n \in \mathbf{Z}$.

(c) Nonzero prime ideals p such that $\theta(p) = p$.

(d) Nonzero prime ideals p such that $\theta^N(p) = p$ for some $N \in \mathbf{Z}$, $N \neq 1$.

For results about part of the left spectrum lying over ideals in (a) above, refer to [R1, Sect. 3.2.5, Chap. 2]. For part of the left spectrum lying over ideals in (c), the idea is to consider $Spec_l(R/p)\{\theta', \xi'\}$ (θ' and ξ' are resp. the natural extension of θ and the image of ξ). Then we have a result (cf. [R1, Sect. 3.2.6, Chap. 2]) that part of the spectrum lying over an ideal from (c) is in bijection with the set of left ideals from $Spec_l(R/p)\{\theta', \xi'\}$ whose intersection with R/p is the zero ideal. For results about part of the spectrum lying over ideals from (d) above, refer to [R1, Theorem 6.9.1, Chap. 4] (understood in ring-theoretic language). Now we state results about part of the spectrum lying over ideals from (b) in the list above.

THEOREM 2.0.1. (i) *Let $p \in SpecR$ be such that $\theta^n(p) \neq p$ for any $n \in \mathbf{Z}$.*

(1) *If $\theta^{-1}(\xi) \in p$, then the left ideal*

$$p_{1,1} = p + R\{\theta, \xi\}x + R\{\theta, \xi\}y$$

is a two-sided ideal from $Spec_l R\{\theta, \xi\}$.

(2) *If $\theta^{-1}(\xi) \in p$, $\theta^i(\xi) \notin p$ for $0 \leq i \leq n-1$, and $\theta^n(\xi) \in p$, then the left ideal*

$$p_{1,n+1} = R\{\theta, \xi\}p + R\{\theta, \xi\}x + R\{\theta, \xi\}y^{n+1}$$

belongs to $Spec_l R\{\theta, \xi\}$.

(3) *If $\theta^i(\xi) \notin p$ for $i \geq 0$ and $\theta^{-1}(\xi) \in p$, then*

$$p_{1,\infty} = R\{\theta, \xi\}p + R\{\theta, \xi\}x$$

belong to $Spec_l R\{\theta, \xi\}$.

(4) *If $\xi \in p$ and $\theta^{-i}(\xi) \notin p$ for $i \geq 1$, then the left ideal*

$$p_{\infty,1} = R\{\theta, \xi\}p + R\{\theta, \xi\}y$$

belongs to $Spec_l R\{\theta, \xi\}$.

(ii) *If the ideal in* 2, 3, *or* 4 *is maximal, then the corresponding left ideal from $Spec_l\{\theta, \xi\}$ is maximal.*

(iii) *An ideal* $\mathbf{p}$ *from $Spec_l R\{\theta, \xi\}$ such that $\theta^\nu(\xi) \in \mathbf{p}$ for some $\nu \in \mathbf{Z}$ is equivalent to one of the left ideals above for a uniquely defined $p \in SpecR$. This means that if p and p' are prime ideals of ring R, and (α, β) and (ν, μ) take values $(1, \infty)$, $(\infty, 1)$, (∞, ∞), or $(1, n)$, then $p_{\alpha,\beta}$ is equivalent to $p'_{\nu,\mu}$ if and only if $\alpha = \nu$, $\beta = \mu$, and $p = p'$.*

This theorem gives part of the spectrum lying over ideals with infinite orbits and containing ξ. Now we state a theorem about part of the spectrum lying over ideals with infinite orbits and not containing ξ. For proofs of both of these theorems, refer to [R1, Chap. 2].

THEOREM 2.0.2. *Let $\approx$ denote the equivalence relation induced by the restriction* (*to $Spec_l R$*) *of the preorder $\succ$ mentioned above.*

(i) *Let p be a prime ideal of ring R such that $\theta^i(\xi) \notin p$ and $\theta^i(p) \neq p$ for all integers i. Then the ideal*

$$p_{\infty,\infty} = R\{\theta, \xi\}p$$

belongs to $Spec_l R\{\theta, \xi\}$.

(ii) *If* $\mathbf{p}$ *is a left ideal in* $R\{\theta, \xi\}$ *such that* $\mathbf{p} \cap R = p$, *then* $\mathbf{p} = p_{\infty,\infty}$. *In particular, if* p *is a maximal ideal, then* $p_{\infty,\infty}$ *is a maximal left ideal.*

(iii) *If a prime ideal* p' *in* R *is such that* $p_{\infty,\infty} \approx p'_{\infty,\infty}$, *then* $p' = \theta^n(p)$ *for some integer* n. *Conversely,* $\theta^n(p)_{\infty,\infty} \approx p_{\infty,\infty}$ *for every* $n \in \mathbf{Z}$.

This approach can be generalized to hyperbolic categories where the role of the automorphism θ is played by an auto-equivalence of the underlying category, and that of the element ξ is played by an endomorphism of the identity functor of the underlying category. For example, in the case of hyperbolic rings the underlying category would be the category *Spec*$(R$-*mod*$)$ and the endomorphism of the identity functor is induced by the action of ξ on each R-module. The advantage of switching to hyperbolic categories is that, for example, we may consider hyperbolic rings over noncommutative rings. There are analogous theorems about the left spectra of hyperbolic categories. (Refer to [R1, Sect. 6, Chap. 4].)

3. CLASSIFICATION THEOREM

In this section we see that the relations of the algebras described in the Introduction force the 7-parameters to satisfy certain conditions. This results in a classification theorem for the 7-parameter family. The result is the following:

THEOREM 3.0.3. *Let* D *be any 7-parameter deformation of* $U(sl_2)$ *described by relations* (1.1) *for which* $\alpha\epsilon\gamma\beta \neq 0$ *or* $\alpha\epsilon\gamma\zeta \neq 0$. *Then* D *must be isomorphic to one of the following three algebras*:

(1) *The* (*conformal* sl_2)*-algebra described by the relations*

$$xz - azx = x, \qquad zy - ayz = y, \qquad yx - cxy = bz^2 + z.$$

(2) *The two double skew polynomial extensions described by the relations*

(a) $xz - zx = x,\ zy - yz = \zeta y,\ yx - \gamma xy = 0$

(b) $xw = \alpha wx,\ wy = \epsilon yw,\ yx = \gamma xy.$

Proof. Recall the relations of the 7-parameter deformations described in (1.1) in the Introduction. There are two possible cases: $\epsilon \neq 1$ and $\epsilon = 1$.

Case 1. $\epsilon \neq 1$. Then we may change the coordinates by $w = z - \zeta/(1-\epsilon)$. This changes the relations as

$$xw = (\alpha w + \beta')x, \qquad wy = y(\epsilon w), \qquad yx - \gamma xy = f(w),$$

where $\beta' = \beta + (\alpha - 1)\zeta/(1 - \epsilon)$ and $f(w)$ is a quadratic in w which is zero iff $\delta = \sigma = 0$. We may compute the commutation relation of w with yx in two different ways:

$$yxw = y(\alpha w + \beta')x = \left(\frac{\alpha}{\epsilon}w + \beta'\right)yx$$

and

$$\begin{aligned} yxw &= (\gamma xy + f(w))w \\ &= \frac{\gamma}{\epsilon}xwy + f(w)w \\ &= \frac{\alpha w + \beta'}{\epsilon}(yx - f(w)) + f(w)w \\ &= \left(\frac{\alpha}{\epsilon}w + \frac{\beta'}{\epsilon}\right)yx - f(w)\left(\left(\frac{\alpha}{\epsilon} - 1\right)w + \frac{\beta'}{\epsilon}\right). \end{aligned}$$

Comparing these two expressions and rearranging the terms gives

$$\beta'\left(\frac{1}{\epsilon} - 1\right)yx = f(w)\left(\left(\frac{\alpha}{\epsilon} - 1\right)w + \frac{\beta'}{\epsilon}\right).$$

If $\beta' \neq 0$, then $yx \in k[w]$. Thus yx commutes with w which gives that $(\alpha w + \beta')/\epsilon = w$. This happens iff $\alpha = \epsilon$ and $\beta' = 0$ which is a contradiction with our assumption.

Now, let $\beta' = 0$. The above relation then reduces to either $\alpha = \epsilon$ or $f(w) = 0$ which is the same as respectively either $\alpha = \epsilon$ or $\delta = \sigma = 0$. If $\alpha = \epsilon$ then since $\beta' = 0$, $\beta = \zeta$. By a standard homothety, we get then this algebra isomorphic to the conformal sl_2 algebra. In the latter case, $f(w) = 0$. This algebra is then isomorphic to the algebra whose relations are described in (2.b) in the statement above.

Case 2. $\epsilon = 1$. Here we may split this case in two subcases: $\alpha = 1$ and $\alpha \neq 1$.

If $\alpha \neq 1$ then we change coordinates by $w = z - \beta/(1 - \alpha)$. It is easy to see that computing yxw in two different ways yields the relation

$$\frac{\zeta'}{\epsilon}(1 - \alpha)yx = f(w)\left(\left(1 - \frac{\alpha}{\epsilon}\right)w + \frac{\zeta'}{\epsilon}\right),$$

where $\zeta' = \zeta + (\epsilon - 1)\beta/(1 - \alpha)$. Exactly as before, we can easily check that $\zeta' = 0$. Thus either $f(w) = 0$ or $\alpha = \epsilon$. Since we are considering the case of $\epsilon = 1$ and $\alpha \neq 1$, we cannot have $\alpha = \epsilon$. The only remaining case is then $f(w) = 0$. This is the algebra (2.b) listed in the proposition.

Thus the only remaining case is then $\alpha = 1$ and $\epsilon = 1$. In the original coordinates, the relations are

$$xz = (z + \beta)x, \qquad zy = y(z + \zeta), \qquad yx - \gamma xy = \delta z^2 + \sigma z.$$

Again we compute zxy in two different ways. This gives the following identity:

$$(\delta z^2 + \sigma z)(\beta - \zeta) = 0.$$

This means that either $\beta = \zeta$ or $\delta = \sigma = 0$. By using certain homotheties, the case $\beta = \zeta$ again gives the conformal sl_2 algebra. For the other case, we may assume that $\beta \neq \zeta$. We use homotheties on β which gives an algebra isomorphic to the one listed in (2)(a) in the proposition. ∎

The algebras listed in the above statement are generically nonisomorphic. The algebras (2)(a) and (2)(b) are double skew polynomial extensions because they may be viewed as skew polynomial extensions of skew polynomial extensions of $k[w]$. This is made explicit in Section 5.

4. THE LEFT SPECTRUM AND THE IRREDUCIBLE REPRESENTATIONS OF THE CONFORMAL sl_2 ALGEBRAS

Key Remark. The conformal sl_2 algebra described in Theorem 3.0.3 of Section 3 is a *hyperbolic ring* $R\{\theta, \xi\}$, where $R = k[z, \xi]$, and $\theta = \theta_{abc}$ is the automorphism of R which sends $f(z, \xi)$ into $f(az + 1, \xi - u(az + 1)/c)$, where $u(z) = bz^2 + z$ and a, b, c are in k. Observe that

$$xr = \theta(r)x, \qquad yr = \theta^{-1}(r)y, \text{ for all } r \in R,$$

and

$$xy = \xi, \qquad yx = \theta^{-1}(\xi).$$

Thus we may apply theorems about the left spectrum of hyperbolic rings to the conformal sl_2 algebras. It is convenient to split the algebras in two families: $a \neq 1$ and $a = 1$.

4.1. *Non-degenerate Case*

Throughout this section $f(w)$ will denote the polynomial $bw^2 + (2b\gamma + 1)w + (b\gamma^2 + \gamma) \in k[w]$.

By non-degenerate we mean the case $a \neq 1$. For convenience, replace z by $w = z - \gamma$, where $\gamma = 1/1 - a$. Thus θ sends $g(w, \xi)$ to $g(aw, (\xi -$

$f(aw))/c)$. Thus, in this case, the algebra $U_{abc}(sl_2)$ can be regarded as a *hyperbolic ring* $R\{\theta, \xi\}$, where $R = k[w, \xi]$. θ^{-1} is described by slightly more convenient formulas:

$$\theta^{-1}(w) = \frac{w}{a}, \qquad \theta^{-1}(\xi) = c\xi + f(w).$$

Since $R = k[w, \xi]$ is a commutative noetherian ring, every left ideal $\mathbf{p}$ in the left spectrum of $R\{\theta, \xi\}$, is equivalent to a left ideal $\mathbf{p}'$ such that $\mathbf{p}' \cap R$ is a prime ideal in R. See [R1, Sect. 3.2.1, Chap 2]. Thus it will be sufficient to study the left ideals lying over prime ideals of $R = k[w, \xi]$. (The notion of lying over was given in Section 2.) The left spectrum will be computed by studying the left ideals lying over prime ideals in R via the following four alternatives: Let $p = \mathbf{p} \cap R$ be a prime ideal in $R = k[w, \xi]$.

(a) p is not invariant under θ^n for any n.

(b) $p = \{0\}$.

(c) p is invariant under θ.

(d) p is invariant under θ^N for some N but $\theta^m(p) \neq p$ for $1 \leq m \leq N-1$.

We consider these cases in the following sections.

4.1.1. *Case* (a). *Infinite Orbits*

An easy inductive computation shows that for $n \geq 1$,

$$\theta^n(\xi) = c^{-n}\left(\xi - \left(ba^2\left(\frac{1-(a^2c)^n}{1-a^2c}\right)w^2 + (2b\gamma+1)a\left(\frac{1-(ac)^n}{1-ac}\right)w + (b\gamma^2+\gamma)\left(\frac{1-c^n}{1-c}\right)\right)\right) \tag{4.1}$$

and

$$\theta^{-n}(\xi) = c^n\xi + \frac{b}{a^{2n-2}}\frac{1-(a^2c)^n}{1-a^2c}w^2 + \frac{(2b\gamma+1)}{a^{n-1}}\frac{1-(ac)^n}{1-ac}w + (b\gamma^2+\gamma)\frac{1-c^n}{1-c}. \tag{4.2}$$

We use Theorem 2.0.1 to compute that part of the spectrum which consists of left ideals $p \in SpecR$ such that $\theta^n(p) \neq p$ for any $n \in \mathbf{Z}$ and $\xi \in \theta^n(p)$ for some $n \in \mathbf{Z}$. For the following five propositions, let $p \in SpecR$ be such that $\theta^n(p) \neq p$ for any $n \in \mathbf{Z}$.

PROPOSITION 4.1.1. *If $\xi \in p$ and $f(w) \in p$, then the left ideal*

$$p_{1,1} = p + R\{\theta, \xi\}x + R\{\theta, \xi\}y$$

is a two-sided ideal from $\mathrm{Spec}_l R\{\theta, \xi\}$. In this case, the ideal $p = (\xi, w + \gamma)$ or $(\xi, (b\gamma + 1)/b)$. The corresponding quotient module is one dimensional.

Proof. Fix a prime ideal p of the ring $R = k[w][\xi]$ and denote by p', the intersection $p \cap A$ where $A = k[w]$. To begin with we make an observation: if $\theta^{-1}(\xi) = c\xi + f(w) \in p$, then the ideal p is generated by $c\xi + f(w)$ and the prime ideal $p' = p \cap A$ of the ring $A = k[w]$:

$$p = k[w][\xi](c\xi + f(w)) + k[w][\xi]p'.$$

Now suppose that $\xi \in p$ and $\theta^{-1}(\xi) \in p$. Then $f(w) \in p$, which implies that $w + \gamma \in p'$ or $w + (b\gamma + 1)/b \in p'$. There are two left maximal ideals in $U_{abc}(sl_2)$ which are generated by $x, y, \xi, (w + \gamma$, or $w + (b\gamma + 1)/b)$. The corresponding irreducible modules are isomorphic to k (as a k-vector space), where the action is given by $x.k = y.k = \xi.k = 0$ and $w.\alpha = -\gamma\alpha$ for $\alpha \in k$ (or $w.\alpha = -((b\gamma + 1)/b)\alpha$) (cf. Theorem 2.0.1 above). ∎

For the remainder of this section, let

$$g_l(w) = b\left(\frac{1 - (a^2c)^{l+1}}{1 - a^2c}\right)w^2 + (2b\gamma + 1)\left(\frac{1 - (ac)^{l+1}}{1 - ac}\right)w + (b\gamma^2 + \gamma)\left(\frac{1 - c^{l+1}}{1 - c}\right)$$

for any integer $l \geq 1$.

PROPOSITION 4.1.2. *If $c\xi + f(w) \in p$, $g_n(w) \in p$, and $g_i(w) \notin p$ for $0 \leq i \leq n - 1$, then the left ideal*

$$p_{1,n+1} = R\{\theta, \xi\}p + R\{\theta, \xi\}x + R\{\theta, \xi\}y^{n+1}$$

belongs to $\mathrm{Spec}_l R\{\theta, \xi\}$. In this case, p is generated by $c\xi + f(w)$ and $h_n(w)$, where $h_n(w)$ is either $g_n(w)$ or a linear factor of $g_n(w)$. The corresponding quotient module is n-dimensional.

Proof. Let $\theta^n(\xi) \in p$, $\theta^i(\xi) \notin p$ for $0 \leq i \leq n - 1$ and $\theta^{-1}(\xi) \in p$. It follows from Eq. (4.1) above that $\theta^n(\xi) \in p$ $(n \geq 1)$ if and only if for

$n \geq 1$, the polynomial

$$\begin{aligned} & b\left(a^2c\left(\frac{1-(a^2c)^n}{1-a^2c}\right)+1\right)w^2+(2b\gamma+1)\left(ac\left(\frac{1-(ac)^n}{1-ac}\right)+1\right)w \\ & \quad +(b\gamma^2+\gamma)\left(c\left(\frac{1-c^n}{1-c}\right)+1\right) \\ & \quad = b\left(\frac{1-(a^2c)^{n+1}}{1-a^2c}\right)w^2+(2b\gamma+1)\left(\frac{1-(ac)^{n+1}}{1-ac}\right)w \\ & \quad\quad +(b\gamma^2+\gamma)\left(\frac{1-c^{n+1}}{1-c}\right) \\ & \quad = g_n(w) \end{aligned}$$

is in p. We may suppose that n is the smallest integer such that $g_n(w) \in p$.

If $g_n(w)$ is irreducible in $k[w]$, then p' is generated by $g_n(w)$ which means that the ideal p is generated by $c\xi + f(w)$ and $g_n(w)$.

If $g_n(w)$ is reducible, then it has two roots in k, say α_n and β_n. If these two are not equal then we get two left maximal ideals generated by x, y^{n+1}, $c\xi + f(w)$, and $(w-\alpha_n$ or $w-\beta_n)$. Observe that $\alpha_n = \beta_n$ iff discriminant of $g_n(w) = 0$. In this case, we get only one left maximal ideal generated by x, y^{n+1}, $c\xi + f(w)$, and $(w-\alpha_n)$. The corresponding irreducible representations are finite dimensional. They are further described in Subsection 4.2. ∎

PROPOSITION 4.1.3. (1) *If $c\xi + f(w) \in p$ and $g_i(w) \notin p$ $\forall i \geq 0$, then*

$$p_{1,\infty} = R\{\theta, \xi\}p + R\{\theta, \xi\}x$$

belongs to $Spec_l R\{\theta, \xi\}$, where $g_i(w)$ is as defined earlier for all $i \geq 1$ and $g_0(w) = f(w)$. In this case, p is generated by $(c\xi + f(w), g(w))$, where $g(w)$ is an irreducible polynomial not dividing $f(w)$ or $g_n(w)$ for any $n \geq 0$. The corresponding quotient modules are infinite dimensional.

(2) *If $\xi \in p$, $f(w) \notin p$, and c is not a root of unity, then*

$$p_{\infty,1} = R\{\theta, \xi\}p + R\{\theta, \xi\}y$$

belongs to $Spec_l R\{\theta, \xi\}$. In this case, the ideal p is generated by $(\xi, g(w))$, where $g(w)$ is any irreducible polynomial not dividing $f(w)$.

Proof. To prove (1) and (2), we define the polynomial

$$g_{-n}(w) = \frac{b}{a^{2n-2}} \frac{1-(a^2c)^n}{1-a^2c} w^2 + \frac{(2b\gamma+1)}{a^{n-1}} \frac{1-(ac)^n}{1-ac} w + (b\gamma^2+\gamma)\frac{1-c^n}{1-c}$$

for each integer $n \geq 1$. Observe that $\theta^i(\xi) \notin p$ for $i \geq 0$, $\theta^{-1}(\xi) \in p$ (resp. $\xi \in p$, $\theta^{-i}(\xi) \notin p$ for $i \geq 1$) iff $c\xi + f(w) \in p$ and $g_i(w) \notin p$ $\forall i \geq 0$ (resp. $\xi \in p$ and $g_{-i}(w) \notin p$). Following Theorem 2.0.1 corresponding to this case, we observe that any irreducible polynomial $g(w) \in k[w]$ such that $g(w)$ does not divide $g_n(w)$ for any $n \geq 0$ (resp. $g_{-n}(w)$) gives the left maximal ideal generated by $(c\xi + f(w), x)$ and $g(w)$ (resp. generated by (ξ, y) and $g(w)$). The requirement about $g(w)$ ensures that $g_i(w) \notin p$ $\forall i \geq 0$ (resp. $g_{-i}(w) \notin p$). The corresponding quotient modules are clearly infinite dimensional representations. ∎

PROPOSITION 4.1.4. *If the ideal p in the last three propositions is maximal, then the corresponding left ideal from* $Spec_l R\{\theta, \xi\}$ *is left maximal.*

Proof. The proof of this proposition follows at once from (ii) of Theorem 2.0.1. ∎

A direct application of (iii) of Theorem 2.0.1 yields the following:

PROPOSITION 4.1.5. *Let* $\mathbf{p}$ *be a left maximal ideal in* $U_{abc}(sl_2)$ *such that* $\theta^m(\xi) \in p = \mathbf{p} \cap k[w, \xi]$ *for some integer m and* $\theta^n(p) \neq p$ *for any integer* n. *Then* $\mathbf{p}$ *is equivalent to one and only one of the left ideals described in Propositions* 4.1.1–4.1.3. *In particular, if* $p \neq p'$, *then any two left ideals lying over p and p′ are inequivalent*.

Observe that in view of the above propositions, we may now restrict to the part of the spectrum consisting of the left ideals whose intersection with R has an infinite orbit under the action of θ and $\xi \notin \theta^n(p)$ for any $n \in \mathbf{Z}$. Any maximal ideal p of $R = k[w, \xi]$ such that $\theta^m(\xi) \notin p$ for any m gives a left maximal ideal $p_{\infty,\infty} = R\{\theta, \xi\}p$. (The formulas for $\theta^m(\xi)$ are given by (4.1) and (4.2) above.) In particular, if k is algebraically closed, then for any $\alpha, \beta \in k$, $\xi - \alpha$ and $w - \beta$ generate a maximal ideal if the expressions

$$c^n\alpha + \frac{b}{a^{2n-2}} \frac{1-(a^2c)^n}{1-a^2c} \beta^2 + \frac{(2b\gamma+1)}{a^{n-1}} \frac{1-(ac)^n}{1-ac} \beta + (b\gamma^2+\gamma)\frac{1-c^n}{1-c}$$

and

$$\alpha - \left(ba^2 \left(\frac{1-(a^2c)^n}{1-a^2c} \right) \beta^2 + (2b\gamma + 1) a \left(\frac{1-(ac)^n}{1-ac} \right) \beta + (b\gamma^2 + \gamma) \left(\frac{1-c^n}{1-c} \right) \right)$$

are both nonzero. Further, for any irreducible polynomial $g(w) \in k[w]$ not dividing $\theta^n(\xi)$ $\forall n \in \mathbf{Z}$ and whose orbit is infinite, we get a left ideal which is not left maximal. These will then correspond to the non-closed points of the left spectrum. (For the definition of closed points in the left spectrum, refer to [R1].)

This together with earlier description exhausts the case of infinite orbits, i.e., any ideal $\mathbf{p}$ such that $\mathbf{p} \cap k[w, \xi]$ lies in an infinite orbit under the action of θ is equivalent to one and only one of the left ideals in Propositions 4.1.1–4.1.3 or the left ideals described in the previous paragraph. This follows at once from Theorems 2.0.1 and 2.0.2.

4.1.2. *Case* (b). *Non-degenerate Part of the Spectrum*

By non-degenerate part of the spectrum we mean the left ideals $\mathbf{p}$ of $R\{\theta, \xi\}$ such that $\mathbf{p} \cap R = \{0\}$. Observe that if $\mathbf{p}$ is a left ideal from $Spec_l R\{\theta, \xi\}$, then $\mathbf{p} \cap R = \{0\} \in SpecR$; which implies that $R - \{0\}$ is a θ-invariant multiplicative subset in the domain R. Thus localizing $R\{\theta, \xi\}$ at $R - \{0\}$ gives the ring $K(R)\{\Theta, \xi'\}$, where $K(R)$ is the field of fractions of the ring R, Θ is the natural extension of the automorphism θ of R, and ξ' is the image of ξ under the localization map. Since localizations respect the left spectrum (cf. Lemma 0.3.1 of [R2]), the localization Q at $R - \{0\}$ sends the left ideal $\mathbf{p}$ into the spectrum of the ring $K(R)\{\Theta, \xi'\}$.

Now since $\xi \neq 0$, ξ' is an invertible element and thus the ring $K(R)\{\Theta, \xi'\}$ is isomorphic to the ring $K(R)[x, x^{-1}; \Theta]$. The ring $K(R)[x, x^{-1}; \Theta]$ is a Laurent skew polynomial ring with coefficients in $K(R)$ and relations are given by

$$x\xi' = \Theta(\xi')x = \frac{\xi' - f(aw)}{c}x$$

and

$$xw = \Theta(w)x = awx$$

(with natural extension for x^{-1}). Recall that the left spectrum of the ring $K(R)[x, x^{-1}; \Theta]$ is isomorphic to that of the ring $K(R)[x, \Theta]$ except for one point: the left ideal generated by x. Furthermore since $K(R)[x, \Theta]$ is a left and right principal domain, any left ideal from the left spectrum is equivalent to a left maximal ideal and any left maximal ideal is generated by an irreducible polynomial $r(x)$ ($\neq \lambda x$ for any $\lambda \in K(R)$). Thus, the left spectrum of $K(R)[x, x^{-1}; \Theta]$ is isomorphic to the set of all (non-equivalent) irreducible skew polynomials in $K(R)[x, \Theta]$ not equal to λx, for any $\lambda \in K(R)^*$.

In general, the part of the spectrum of $R\{\theta, \xi\}$ lying over $\{0\}$ embeds in the left spectrum of $K(R)[x, x^{-1}; \Theta]$ and is not an isomorphism. We can however determine whether a particular element in the left spectrum of $K(R)[x, x^{-1}; \Theta]$ is in the image of this embedding:

Let $\mathbf{p}$ be an element in $Spec_l K(R)[x, x^{-1}; \Theta]$ and let $M_{\mathbf{p}}$ be the corresponding quotient $(K(R)[x, x^{-1}; \Theta])$-module. Let $M'_{\mathbf{p}}$ denote the same module viewed as a $R\{\theta, \xi\}$-module. If $\mathrm{Ass}(M'_{\mathbf{p}})$ is non-empty, then $\mathbf{p}$ is in the image of the embedding mentioned above. Further, if $\mathrm{Soc}(M'_{\mathbf{p}})$ is nonzero, then $M'_{\mathbf{p}}$ is an irreducible module.

In our case, $R = k[w, \xi]$ and hence $K(R) = k(w, \xi)$, the field of rational functions in w and ξ. Two skew polynomials are equivalent iff the right ideal generated by one of them intersects non-trivially with the left ideal generated by the other. In general, it is hard to know the set of irreducible polynomials modulo this equivalence. Summing up, we have the following proposition:

PROPOSITION 4.1.6. *The set of left ideals in $Spec_l R\{\theta, \xi\}$, whose intersection with R is* (0), *embeds in the left spectrum of $k(w, \xi)[x, x^{-1}; \Theta]$ which in turn is in bijection with the set of* (*non-equivalent*) *irreducible polynomials in $k(w, \xi)[x, \Theta]$* ($\neq \lambda x$ *for any* $\lambda \in k(w, \xi)$), *where Θ is the natural extension of the automorphism of R to that of $k(w, \xi)$.*

4.1.3. *Case* (c). *Orbits of θ-Invariant Ideals*

Now we describe left ideals $\mathbf{p}$ of the left spectrum lying over θ-invariant ideals of $R = k[w, \xi]$. Suppose that $\mathbf{p}$ is a left ideal from the left spectrum of $R\{\theta, \xi\}$ such that $\mathbf{p} \cap R = p$ is a θ-invariant prime ideal of R. Then θ induces an automorphism θ' of the quotient ring $R/p = R'$; and the canonical map $\pi' : R \to R'$ extends to a ring homomorphism $\pi : R\{\theta, \xi\} \to R'\{\theta', \xi'\}$ where $\xi' = \pi(\xi)$, $\pi'(x) = x$, $\pi'(y) = y$. Since π' is an epimorphism, the image $\mathbf{p}'$ of the left ideal $\mathbf{p}$ belongs to the left spectrum, and $\mathbf{p} \cap R' = \{0\}$. There are two possibilities: either $\xi \in p$ or $\xi \notin p$.

(a) Suppose $\xi \in p$. Since p is θ-invariant, $\theta^{-1}(\xi) \in p$. Thus, both xy and yx are in p. Using this data, it is easy to show that the left spectrum of $R'\{\theta', \xi' = 0\}$ is the union of three disjoint sets: (1) The set of left ideals in

$Spec_l R'\{\theta', \xi'\}$ containing both x and y. This set is closed in the left spectrum and is isomorphic to $Spec R'$. (2) The set of left ideals in $Spec_l R'\{\theta', \xi'\}$ containing x but not y. This is an open set in the left spectrum and is isomorphic to $Spec_l R'[x, x^{-1}; \theta']$. (3) The set of left ideals $Spec_l R'\{\theta', \xi'\}$ containing y but not x. This is an open set in the left spectrum and is isomorphic to $Spec_l R'[y, y^{-1}; \theta']$. (For a discussion of topologies on the left spectrum, refer to [R1].)

(b) Suppose $\xi \notin p$. Then, it is easy to show that the set of left ideals in the left spectrum of $R\{\theta, \xi\}$ which do not contain ξ and whose intersection with R are θ-invariant ideals embeds in $Spec_l K(R')[x, x^{-1}; \Theta']$ (because localizations respect the left spectrum). Here Θ' is the natural extension of θ' to $K(R')$.

PROPOSITION 4.1.7. *Let p be a θ-invariant ideal in $Spec R$ and $\xi \in p$. Then the set of left ideals from $Spec_l R\{\theta, \xi\}$ lying over p is the disjoint union of the following two subsets: The left ideals*

$$\mathbf{p} = R\{\theta, \xi\}p + R\{\theta, \xi\}x + R\{\theta, \xi\}g(y)$$

or

$$\mathbf{p} = R\{\theta, \xi\}p + R\{\theta, \xi\}h(x) + R\{\theta, \xi\}y,$$

where $h(x)$ and $g(y)$ are irreducible polynomials in $k[x]$ and $k[y]$ resp., not vanishing at zero. In this case, $p = (\xi, w + \gamma)$ or $(\xi, w + (b\gamma + 1)/b)$.

Proof. Since $\xi \in p$ and p is a θ-invariant ideal, $\theta^{-1}(\xi) \in p$. Thus $f(w) \in p$. Observe that $f(w) = (w + \gamma)(bw + (b\gamma + 1))$. Thus $p = (\xi, w + \gamma)$ or $(\xi, bw + (b\gamma + 1))$. In either case, $R' = R/p \cong k$. Thus by the remarks made in this section, the set of left ideals lying over such p is in bijection with the union of $Spec_l k[x, x^{-1}; \Theta] = Spec k[x] - \{(x)\}$ and $Spec_l k[y, y^{-1}; \Theta] = Spec k[y] - \{(y)\}$. Each left ideal in these spectra is equivalent to a left ideal generated by an irreducible polynomial in $k[x]$ or $k[y]$ not vanishing at zero. ∎

PROPOSITION 4.1.8. *Let p be a θ-invariant ideal in $Spec R$ and $\xi \notin p$. Then part of the left spectrum lying over p embeds in the union of following sets:*

(b1) *Left ideals from $Spec_l k(\xi)[x, x^{-1}; \Theta]$.*

(b2) *Left ideals from $Spec_l (K[\xi]/(h(\xi)))[x, x^{-1}; \Theta]$, where $K = (k[w]/(g))$ for some irreducible polynomial $g(w) \in k[w]$ and $h(\xi)$ is an irreducible polynomial in $K[\xi]$ such that $p = (g(w)$, preimage of $h(\xi)$ in $k[\xi])$.*

(b3) *Left ideals from $Spec_l (k(\xi)[w]/g(w))[x, x^{-1}; \Theta]$, where $g(w) \in k(\xi)[w]$ is an irreducible polynomial.*

(b4) *Left ideals from* $Spec_l(k(w)[\xi]/h(\xi))[x, x^{-1}; \Theta]$, *where* $h(\xi) \in k(w)[\xi]$ *is an irreducible polynomial.*

Here the automorphism Θ *in each case is the natural extension of the automorphism* θ.

Remark. Note that the embedding in the statement is, in general, not an isomorphism.

Proof. Let p be a prime ideal of $R = k[w, \xi]$ such that $\theta(p) = p$ and $\xi \notin p$. Let $p' = p \cap k[w]$. Then p' is a θ-invariant prime ideal of $k[w]$. Let it be generated by an irreducible monic polynomial $g(w)$. Then $\theta(p') = p'$ is equivalent to the condition $g(w) = a^m g(w)$ where $m = \deg g(w)$. Thus there are the following three possibilities:

(1) $g(w) = w$,

(2) a is a lth-root of unity and $g(w) = \Phi(w^l)$ for any polynomial Φ such that g is irreducible,

(3) $g(w) = 0$.

We analyze each of these cases.

Case 1. Let $g(w) = w$. Clearly, $p = Rw \in SpecR$ is a θ-invariant ideal. Then $R' \cong R/p = k[w, \xi]/(w) \cong k[\xi]$. Since p is a nonzero ideal and $\xi \notin p$, the set of left ideals from the left spectrum lying over $p = Rw$ embeds in $Spec_l K(R')[x, x^{-1}; \Theta] = Spec_l k(\xi)[x, x^{-1}; \Theta]$. (Here Θ is the automorphism of $K(R')$ induced from θ.) Any left ideal in the left spectrum of $K(R')[x, x^{-1}; \Theta]$ is equivalent to the left ideal generated by a skew irreducible polynomial g (with coefficients from the field of rational functions $k(\xi)$) not equal to λx, $\lambda \in k(\xi)^*$. This is the case (b1) of the statement.

Thus, we may now assume that $p \neq Rw$. Then, consider the ring $R/(w) \cong k[\xi]$. The induced automorphism θ' acts on $k[\xi]$ by $\theta'(\xi) = (\xi - (b\gamma^2 + \gamma))/c$. Let the image of p in $k[\xi]$ be generated by an irreducible polynomial $h(\xi)$. There are two possibilities: $c = 1$ or $c \neq 1$.

If $c = 1$, then $\theta'(\xi) = \xi - (b\gamma^2 + \gamma)$. If $b\gamma^2 + \gamma \neq 0$ then the ideal $k[\xi]h(\xi)$ is θ'-stable iff char $k = p > 0$, $h(\xi) = \psi(\xi^p)$, and $h(-(b\gamma^2 + \gamma)) = 0$. In this case, $p = (w, h(\xi))$. The set of ideals lying over $p(\neq Rw)$ is isomorphic to $Spec_l K(R')[x, x^{-1}; \Theta]$, where $K(R') = k[\xi]/(h(\xi))$ and Θ is the natural extension of the automorphism θ to $K(R')$. Every left ideal of $K(R')$ in the left spectrum of this ring is equivalent to the left ideal generated by a skew irreducible polynomial not equal to λx for any $\lambda \in K(R')^*$.

If $b\gamma^2 + \gamma = 0$ (this is equivalent to the condition $a - b = 1$), then $\theta'(h(\xi)) = h(\xi)$. Since $\xi \notin p$, the description is similar to the above for some irreducible polynomial $h(\xi)$.

Both of these give the case (b2) in the statement of the theorem for $K = k$ and the polynomial $g = w$.

If $c \neq 1$, then let $\alpha = (b\gamma^2 + \gamma)/(1 - c)$. Observe that

$$\theta'(\xi - \alpha) = \frac{\xi - \alpha}{c}.$$

Thus, there are two possibilities for $h(\xi)$: 1) $k[\xi]h(\xi) = k[\xi](\xi - \alpha)$ 2) c is an nth root of 1, and $h(\xi) = \Phi((\xi - \alpha)^n)$ for some polynomial Φ. In either case, the part of the left spectrum lying over ideals generated by w and $h(\xi)$ is isomorphic to $Spec_l K[x, x^{-1}; \theta]$. Here $k = k[\xi]/h(\xi)$. This also corresponds to the case (b2) in the statement.

Case 2. Let a be an lth root of unity and $g = \Phi(w^l)$ for some polynomial Φ. First assume that $p = Rg$. Then we may localize at $k[\xi] - \{0\}$ to get that the set of left ideals lying over p embeds in $Spec_l(k(\xi)[w]/g(w))[x, x^{-1}; \Theta]$. This gives the case (b3) in the proposition.

Now we assume that $p \neq Rg$. Let $\bar{p}$ be the image of p in $R/Rg \cong K[\xi]$, where K is the quotient field $k[w]/(g)$. K is an m-dimensional vector space, $m = \deg g(w)$, with the basis $\overline{w}^s$, $0 \leq s \leq m - 1$. Here, the induced automorphism θ' acts on both K (by sending $\overline{w}^s$ to $a^s\overline{w}^s$) and ξ (by sending ξ to $(\xi - f(a\overline{w}))/c$). Let the image of $\bar{p}$ be generated by an irreducible polynomial $h(\xi)$.

If $a^i c \neq 1$ for $i = 0$, 1, or 2, then it is easy to see that there exists $\alpha \in K$ such that $\theta'(\xi - \alpha) = (\xi - \alpha)/c$. In fact,

$$\alpha = \frac{a^2 b}{1 - a^2 c} w^2 + \frac{a(2b\gamma)}{1 - ac} w + \frac{b\gamma^2 + \gamma}{1 - c}.$$

On the other hand, if $a^i c = 1$ for $i = 0/1/2$, then we make a change of variable, ξ to $\zeta = w^i\xi$, where i is such that $a^i c = 1$. Then $\theta'(\zeta) = \zeta - \alpha$ for some $\alpha \in K$.

In either case, the ideal p is generated by $g(w)$ and preimage of $h(\xi)$ in $k[\xi]$. The set of left ideals lying over such p embeds in $Spec_l(K[\xi]/h(\xi))[x, x^{-1}; \Theta]$, where Θ is the natural extension of the automorphism θ. These give the case (b2) in the proposition.

Case 3. $g = 0$. We may assume that $p \neq 0$. (The case when $p = 0$ is already covered.)

We localize R at $k[w] - \{0\}$, and consider the image of p in $K[\xi]$ where $K = k(w)$. Let this image be generated by $h(\xi)$. Observe that the induced automorphism acts non-trivially on the coefficient field $k(w)$. The set of left ideals lying over such p embeds in $Spec_l(k(w)[\xi]/h(\xi))[x, x^{-1}; \Theta]$. This corresponds to the case (b4) of the proposition. ∎

4.1.4. *Case* (d). *Finite Orbits*

To complete the description of the left spectrum in the non-degenerate case, we will describe the part of the spectrum lying over left ideals whose orbits under the action of θ are finite, but not stable. We use [R1, Theorem 6.9.1, Chap. 4] suitably understood in the ring theoretic language.

Fix a positive integer $N > 1$. Let p be a prime ideal in R such that $\theta^N(p) \approx p$, but $\theta^i(p)$ is not equivalent to p if $1 \leq i < N$. We shall compute the representations corresponding to left ideals in $Spec_l R\{\theta, \xi\}$ lying over such an ideal. These will be described in terms of representations of another hyperbolic algebra. We construct a sub-algebra of $R\{\theta, \xi\}$ which will again be hyperbolic (it may not be a hyperbolic sub-algebra though). Denote by φ the automorphism θ^N. Since

$$\varphi(w) = a^N w$$

and

$$\varphi(\xi) = \frac{\xi - p_N(w)}{c^N},$$

where

$$p_N(w) = c^{-N}\left(\xi - \left(ba^2\left(\frac{1-(a^2c)^N}{1-a^2c}\right)w^2 + (2b\gamma+1)a\left(\frac{1-(ac)^N}{1-ac}\right)w + (b\gamma^2+\gamma)\left(\frac{1-c^N}{1-c}\right)\right)\right) \tag{4.3}$$

we may compute θ^N-invariant prime ideals in $SpecR$ in a way similar to the case of θ-invariant prime ideals. Thus the ideal $p' = k[w] \cap p$ can be generated by an irreducible polynomial $g(w)$ with the following possibilities:

(1) $g = \mathrm{w}$

(2) $g = 0$

(3) a^N is lth root of 1 and $g = r(w^l)$ for some polynomial $r \in k[w]$.

Consider the image of the ideal p in R/p'. Let $x' = x^N$ and $y' = y^N$. Then it is easily seen that the ring $R\{\varphi, \zeta\}$ is hyperbolic, where $\zeta = x^N y^N$ and $R = k[w, \xi]$. We need to express the element ζ in terms of ξ. We have a formula

$$\zeta = \xi\theta(\xi)\ldots\theta^{N-1}(\xi).$$

Also, the action of φ is given by the formulas

$$\varphi(w) = a^N w$$

and

$$\varphi(\zeta) = \frac{\xi - p_N(w)}{c^N} \cdots \frac{\xi - p_{2N-1}(w)}{c^{2N-1}},$$

where $p_i(w)$ is the polynomial as in (4.3).

Now using results of the previous section (Proposition 4.1.7–4.1.8), we get the following proposition:

PROPOSITION 4.1.9. *Let p be a $\varphi(=\theta^N)$-invariant ideal in SpecR and suppose that $\theta^i(\xi) \in p$ for some integer i. Let $R' = R/p$ and φ' be the natural automorphism of R' induced from φ. Then the set of left ideals from $Spec_l R\{\varphi, \zeta\}$ lying over p embeds in the disjoint union of following three subsets*:

(a1) *$SpecR'$*

(a2) *$Spec_l K(R')[x', x'^{-1}; \Phi']$*

(a3) *$Spec_l K(R')[y', y'^{-1}; \Phi'^{-1}]$.*

Note that the left ideals in (a2) *and* (a3) *are resp. equivalent to left ideals generated by $h(x')$ and $g(y')$ not vanishing at zero. Also, in this case, p is any prime ideal which contains $\theta^{nN+i}(\xi)$ for all integers n. Here Φ' is the natural extension of φ' to $K(R')$.*

PROPOSITION 4.1.10. *Let p be a $\varphi(=\theta^N)$-invariant ideal in SpecR and suppose that $\theta^i(\xi) \notin p$ for any integer i. Then part of the left spectrum lying over p embeds in the union of following sets*:

(b1) *Left ideals from $Spec_l k(\xi)[x', x'^{-1}; \Theta]$.*

(b2) *Left ideals from $Spec_l(K[\xi]/h(\xi))[x', x'^{-1}; \Theta]$, where $K = (k[w]/(g))$ for some irreducible polynomial $g(w) \in k[w]$ and $h(\xi)$ is an irreducible polynomial in $K[\xi]$ such that $p = g(w)$, preimage of $h(\xi)$ in $k[\xi]$).*

(b3) *Left ideals from $Spec_l(k(\xi)[w]/g(w))[x', x'^{-1}; \Theta]$, where $g(w) \in k(\xi)[w]$ is an irreducible polynomial.*

(b4) *Left ideals from $Spec_l(k(w)[\xi]/h(\xi))[x', x'^{-1}; \Theta]$, where $h(\xi) \in k(w)[\xi]$ is an irreducible polynomial.*

Here the automorphism Θ in each case is the natural extension of the automorphism θ.

Remark. The proofs of the above propositions follow directly from the statements and proofs of Proposition 4.1.7 and 4.1.8 in the last section. We only need to observe that the condition $\zeta \in p$ is equivalent to the condition $\theta^i(\xi) \in p$ for some integer i.

Assume that $\mathbf{p}$ is in $Spec_l R\{\varphi, \zeta\}$ and its intersection with R is a φ-invariant ideal (and that $p = \mathbf{p} \cap R$ is not invariant under θ^i for any $1 \leq i < N$). Let V be a $R\{\varphi, \zeta\}$-module and $V \approx R\{\varphi, \zeta\}/\mathbf{p}$. Observe that V is equivalent to a representation obtained from the above list. Then the following two propositions give a list of representations of $R\{\theta, \xi\}$ obtained from θ^N-invariant prime ideals of R.

PROPOSITION 4.1.11. *Suppose that $\xi \notin \theta^i(p)$ for $1 \leq i < N$. (This corresponds to the case of Proposition* 4.1.10.) *Then every nonzero submodule of $R\{\theta, \xi\} \otimes_{R\{\varphi, \zeta\}} V$ contains $R\{\theta, \xi\} \otimes_{R\{\varphi, \zeta\}} W$ for some $R\{\varphi, \zeta\}$-module W. In particular, $R\{\theta, \xi\} \otimes_{R\{\varphi, \zeta\}} V$ is simple if V is simple.*

PROPOSITION 4.1.12. *Suppose that $\xi \in \theta^i(p)$ for some i; and let l, m be integers such that $1 \leq l \leq m < N$, $\xi \in \theta^l(p)$, $\xi \in \theta^m(p)$, but $\xi \notin \theta^i(p)$ if $0 \leq i < l$, $m < i < N$. (This corresponds to Proposition* 4.1.9.) *Then*

(b1) *$\bigoplus_{l \leq i \leq N} \theta^i(V)$ is a submodule of $R\{\theta, \xi\} \otimes_{R\{\varphi, \zeta\}} V$ (which is isomorphic to $\bigoplus_{0 \leq i < N} \theta^i(V)$), and the quotient module $V_{1,l}$ is isomorphic to $\bigoplus_{0 \leq i < l} \theta^i(V)$). Also, $V_{1,l} \in Spec_l R\{\theta, \xi\}$.*

(b2) *$\bigoplus_{m \leq i < N} \theta^i(V)$ is a submodule of $R\{\theta, \xi\} \otimes_{R\{\varphi, \zeta\}} V$ which is also in $Spec_l R\{\theta, \xi\}$.*

(b3) *If $\mathbf{p}$ is a left maximal ideal in $R\{\varphi, \zeta\}$, then the $R\{\theta, \xi\}$ module $V_{1,l}$ and the module of* (b2) *are simple.*

Remark. The proofs follow at once from [R1, Theorem 6.9.1, Chap. 4]. In the statement of that theorem, $\mathscr{F}$ corresponds to $R\{\theta, \xi\}$ and $\mathscr{G}$ corresponds to $R\{\varphi, \zeta\}$.

The propositions proved in these sections complete the list of all the irreducible representations of $Spec_l R\{\theta, \xi\}$ in the non-degenerate case $a \neq 1$.

4.2. *Finite Dimensional Irreducible Representations*

Throughout this section, it is assumed that k is an algebraically closed field with characteristic 0.

Remark. In our case, the ring R is $k[w, \xi]$. For any prime ideal p in R, if the quotient R/p is a finite dimensional vector space over k, then it must be isomorphic to k and hence p must be a maximal ideal in R. So p

must be of the form $p = (w - \mu, \xi - \lambda)$ for some $\mu, \lambda \in k$. Now suppose that $\mathbf{p}$ is a left maximal ideal of $R\{\theta, \xi\}$ such that $R\{\theta, \xi\}/\mathbf{p}$ is a finite dimensional vector space over k. Since two irreducible modules are equivalent iff they are isomorphic, we may assume that $p = \mathbf{p} \cap R$ is a prime ideal in R. Since $R/p \hookrightarrow R\{\theta, \xi\}/\mathbf{p}$, p must be a maximal ideal in R. We have the following theorem:

THEOREM 4.2.1. *For generic values of $a, b, c \in k$, there are at most two n-dimensional irreducible representations of $U_{abc}(sl_2)$ in each dimension.*

Proof. The idea is to extract finite dimensional representations from each of the cases considered in previous sections. The remark above shows that if $\mathbf{p}$ gives a finite dimensional representation, then $p = \mathbf{p} \cap R$ must have the form $p = (w - \mu, \xi - \lambda)$. Thus, in particular, we will not get any finite dimensional representation from the non-degenerate case (b) of Subsection 4.1.2.

First consider the case (a) (Subsection 4.1.1) of infinite orbits. Since $g_n(w)$ is not irreducible, it has two roots in k, α_n and β_n. Assume that neither α_n nor β_n is a root of $g_i(w)$ for $0 \le i \le n-1$. If these two roots are not equal, then the two ideals $(c\xi + f(\alpha_n), w - \alpha_n)$ and $(c\xi + f(\beta_n), w - \beta_n)$ give two non-equivalent left ideals in the spectrum (cf. Proposition 4.1.2). In this case, we get two left maximal ideals generated by x, y^{n+1}, $c\xi + f(w)$, and ($w - \alpha_n$ or $w - \beta_n$) and thus two irreducible representations in dimension $(n+1)$ which are isomorphic to $\bigoplus_{i=0}^{n} k.y^i \cong U_{abc}(sl_2)/\mathbf{p}$, where the action of x, y, w is natural. Observe that the condition $\alpha_n = \beta_n$ is equivalent to discriminant of $g_n(w)$ being 0. In this case, we get only one maximal ideal generated by x, y^{n+1}, $c\xi + f(w)$ and $(w - \alpha_n)$. For this value of n we get only one irreducible representation. The parameters a, b, and c satisfying (discriminant of $g_n(w)$) $= 0$ thus give at most one irreducible representation in dimension n.

This proves that for any set of parameters $a, b, c \in k$, there are at most two n-dimensional representations in each dimension for $n \ge 2$ obtained via case (a) of Subsection 4.1.1. To prove the theorem it is sufficient to prove the following claim:

CLAIM. *For generic values of $a, b, c \in k$, there are no finite dimensional representations obtained from θ^n-invariant ideals for $n \ge 1$.*

Proof of the Claim. First we consider θ-invariant ideals of R.

If $c \neq 1$, then $p = (\xi - (b\gamma^2 + \gamma)/(1-c), w)$.

If $c = 1$, then $p = (\xi, w)$. (Here we require that $b\gamma^2 + \gamma = 0$, which is same as $a - b = 1$.)

In both the cases, the ring $(R/p)\{\theta, \xi\}$ is isomorphic to $k[x, x^{-1}; \Theta] = k[x, x^{-1}]$. ($\Theta$ is the automorphism of R/p induced from the automorphism θ of R.) The left spectrum of this ring consists of only one dimensional representations. Hence, in both the cases, the corresponding representations of $R\{\theta, \xi\}$ are one dimensional.

Next consider θ^n-invariant ideals for $n > 1$. Since $p = (\xi - \mu, w - \lambda)$ for some $\mu, \lambda \in k$, computing $\theta^N(p)$ for $N > 1$, we get the possibilities for p as follows.

Let c be an Nth root of 1 in k and $a^N \neq 1$. Then $p = (\xi - \lambda, w)$ $\forall \lambda \neq (b\gamma^2 + \gamma)/(1 - c)$.

Let a be an Nth root of 1 in k and $c^N \neq 1$. Then $p = (\xi - \lambda, w - \mu)$ $\forall \mu$ and $\lambda = p_N(\mu)/(1 - c^N)$, where $p_N(w)$ is the polynomial as in Subsection 4.1.4.

Let $a^N = c^N = 1$ and at least one of a or c be an Nth root of 1. Then $p = (w - \mu, \xi - \lambda)$ $\forall \mu, \lambda \in k$.

Thus for generic values of $a, b, c \in k$, there are no finite dimensional representations obtained from the case of finite orbits. ∎

Remark. Note that in the above theorem we do not single out the finite dimensional representations for the case when either a or c is not a root of 1. For all other cases the above result holds true. It should be noted that a similar analysis can produce a list of all finite dimensional representations. We illustrate it for one of the cases in the proof of the proposition above.

Let $N > 1$, c be Nth root of 1 in k and $a^N \neq 1$. Then $p = (\xi - \lambda, w)$ is a θ^N-invariant ideal of R for all λ not equal to $(b\gamma^2 + \gamma)/(1 - c)$. Note that there are no ideals with infinite orbits containing $\theta^{-1}(\xi)$ and $\theta^n(\xi)$ (for some n) for these values of parameters. Also, it is easy to see that there are no θ^i-invariant ideals for $i \neq N$. Thus, all the finite dimensional representations must come from θ^N-invariant ideals listed above.

Now fix an ideal $p = (\xi - \lambda, w)$ with λ as above. The left ideals from the left spectrum of $R\{\varphi, \zeta\}$ (cf. Subsection 4.1.4) which lie over this prime ideal can be listed as follows:

(a) If $\lambda = (b\gamma^2 + \gamma)((1 - c^{\mathrm{i}})/(1 - c))$ for some $i \neq 0$, then

(1) the left ideals generated by $(p, x', y' - \mu)$ for all $\mu \neq 0$

(2) the left ideals generated by $(p, x' - \nu, y')$ for all $\nu \neq 0$.

(b) If $\lambda \neq (b\gamma^2 + \gamma)(1 - c^i/1 - c)$ for all $i \neq 0$, then the left ideals generated by $(p, (x' - \lambda)/n, y' - \eta)$ for $\eta \neq 0$.

To get representations of $R\{\theta, \xi\}$ from this list, we use Propositions 4.1.11 and 4.1.12.

Corresponding to the left ideals from (b) above, the representations are $R\{\theta, \xi\} \otimes_{R\{\varphi, \zeta\}} (R\{\varphi, \zeta\}/\mathbf{p})$. These are all N-dimensional representations.

Corresponding to the left ideals $\mathbf{p}$ lying over ideals from (a) above, there is some n such that $\lambda = (b\gamma^2 + \gamma)((1 - c^n)/(1 - c))$. Note that such an n is unique. Let $V = R\{\phi, \zeta\}/\mathbf{p}$. Then $V_{1,l} = \bigoplus_{0 \le i < n} \theta^i(V)$ and $\bigoplus_{n \le i < N} \theta^i(V)$ are both in the left spectrum. The dimensions of these representations are resp. n and $N - n$.

By previous arguments, this is a complete list of finite dimensional representations for these values of parameters. Similar arguments produce a list in the other non-generic case, i.e., $a^N = 1$, $c^N \neq 1$. This is left to the reader.

4.3. *Degenerate Case*

Degenerate means the case $a = 1$. Recall that we have a hyperbolic ring $R\{\theta, \xi\}$ where $R = k[z, \xi]$. The action of θ is given by $\theta(z) = z + 1$ and $\theta(\xi) = \xi - (bz^2 + z)/c$. For $n \geq 1$,

$$\theta^n(z) = z + (n + 1)$$

and

$$c^n\theta^n(\xi) = \xi - g(z),$$

where

$$g(z) = \xi - \left(\frac{-n^2c^{n+2} + (2n^2 + 2n - 1)c^{n+1}}{(1 - c)^3} b - \frac{(n + 1)^2 c^n + c + 1}{(1 - c)^3} b + \frac{nc^{n+1} - (n + 1)c^n + 1}{(1 - c)^2} \times (2bz + 1) + \frac{1 - c^{n+1}}{1 - c} (bz^2 + z) \right).$$

We may give a similar formula for action of θ^{-n} on $k[z, \xi]$ for $n \geq 1$. Observe that the computations illustrated in the previous sections depend on two facts: The action of θ^n on w is by scalar multiplication and the action of θ^n on ξ is polynomial, linear in ξ and quadratic in w.

In the degenerate case, the action of θ^n on ξ gives a polynomial which is linear in ξ, quadratic in z, and on z is given by translation. If p is the ideal $\mathbf{p} \cap R$, then the important difference from the previous case arises in the possibilities for $p' = p \cap k[z]$. However, that doesn't change the final results about finite dimensional representations. The interested reader may use the above formulas to compute the actual list of representations.

5. SPORADIC CASES

In this section we consider the cases which appear in Case 2 of Theorem 3.0.3. The algebras described there have the relations

$$xz - zx = x, \qquad zy - yz = \zeta y, \qquad yx - \gamma xy = 0 \tag{5.1}$$

$$wx = x(\alpha w), \qquad wy = y\left(\frac{w}{\epsilon}\right), \qquad xy = \gamma yx. \tag{5.2}$$

First we have the following lemma:

LEMMA 5.0.1. *The algebras described by relations* (5.1) *and* (5.2) *above are double skew polynomial extensions of a commutative noetherian ring, i.e., they are of the form* $B[x; \theta_2]$ *where* $B = A[y; \theta_1]$ *for some commutative polynomial ring* A.

Proof. Recall the relations given by (5.1) above. Let $A = k[z]$ and θ_1 be its automorphism given by $\theta_1(z) = z - \zeta$. Then $B = A[y; \theta_1]$ is a skew polynomial ring described by the relation

$$yz - zy = \zeta y.$$

Let θ_2 be the automorphism of A given by $\theta_2(y) = (1/\gamma)y$, $\theta_2(z) = z + 1$. Then the ring $B[x; \theta_2]$ is a skew polynomial extension of A which is isomorphic to the algebra described by the above relations.

Next consider the relations given by (5.2) above. Here let $A = k[w]$ and θ_1 be its automorphism given by $\theta_1(w) = \epsilon w$. Then $B = A[y; \theta_1]$ is a skew polynomial ring described by the relation

$$wy = y\left(\frac{w}{\epsilon}\right).$$

Let θ_2 be an automorphism of A given by $\theta_2(w) = w/\alpha$ and $\theta_2(y) = y/\alpha$. Then the ring $B[x; \theta_2]$ is a skew polynomial extension of A which is isomorphic to the algebra described by (5.2). ∎

For the remainder of the section we assume that k is an algebraically closed field of characteristic zero and that ϵ *is not a root of unity.*

We will describe the left spectrum of these two algebras by using their iterated skew polynomial structure. Thus first we need to study the left spectrum of the base skew polynomial ring. Observe that we may assume that $\epsilon \neq 1$ and $\zeta \neq 0$, since these cases correspond to a skew polynomial extension of a commutative polynomial ring in two variables and the left spectrum of such rings has been computed [R1]. Further, by the above assumptions on k and ϵ, there are no θ^n-invariant ideals in $k[w]$ for $n > 1$.

LEMMA 5.0.2. $U_2(k,\zeta)$: *Let* $\mathbf{p}$ *be a left ideal in the left spectrum of* $B = A[y;\theta_1]$ *where* $A = k[z]$. *Let* $p = \mathbf{p} \cap k[z]$. *Then* $\mathbf{p}$ *is equivalent to one of the following*:

(1) *Left ideals from* $Spec_l k(z)[y, y^{-1}; \Theta_1]$ *under the canonical localization* $k[z][y,\theta_1] \hookrightarrow k(z)[y,\Theta_1]$ (Θ_1 *is the natural extension of* θ_1 *to* $k(z)$).

(2) *Left ideals generated by* $(z - a)$ *for some* $a \in k$.

(3) *Left ideals generated by* y *and* $(z - a)$ *for some* $a \in k$.

(4) *Left ideals generated by* $(y - b)$ *for some* $b \in k$.

(5) *Generic point* $\{0\}$.

Proof. This follows from results of [R1, Chap. 2] applied to $U_2(k,\zeta)$, which is a skew polynomial ring. ∎

LEMMA 5.0.3. *Quantum Plane*: *Let* $\mathbf{p}$ *be a left ideal in the left spectrum of* $B = A[y;\theta_1]$ *where* $A = k[w]$. *Let* $p = \mathbf{p} \cap k[w]$. *Then* $\mathbf{p}$ *is equivalent to one of the following*:

(1) *Left ideals from* $Spec_l k(w)[y, y^{-1}; \Theta_1]$ *under the canonical localization* $k[w][y,\theta_1] \hookrightarrow k(w)[y,\Theta_1]$ (Θ_1 *is the natural extension of* θ_1 *to* $k(w)$).

(2) *Left ideals generated by* $(w - a)$ *for some* $a \in k$.

(3) *Left ideals generated by* $(y - b, w)$ *for some* $b \in k$.

(4) *Left ideals generated by* $(y, w - a)$ *for some* $a \in k$.

(5) *Generic point* $\{0\}$.

Proof. This follows from results of [R1, Chap. 2], applied to the Quantum Plane, which is a hyperbolic ring. ∎

Recall that we are interested in the left spectrum of $B[x,\theta_2]$, where B and θ_2 are as described earlier. Using [R1, Proposition 4.7.1, Chap. 4] and the discussion preceding it, we have the following proposition.

PROPOSITION 5.0.1. *Let* $R = B[x,\theta_2]$, *where* B *and* θ_2 *are as described earlier*. *Then* $Spec_l R$ *is the disjoint union of the following two subsets*:

(1) $Spec_l B$ (*that is, the left ideals generated by left ideals from* $Spec_l B$ *and* x).

(2) $Spec_l B[x, x^{-1}; \theta_2]$ (*that is, specializations of left ideals from this set*).

Remark. To compute part of the spectrum in (2) above, we will treat $Spec_l B[x, x^{-1}; \theta_2]$ as a hyperbolic ring. By [R1, Example 5.1, Chap. 4], the category of their representations is hyperbolic. We will exploit this fact by using theorems about classification of the left spectrum of a hyperbolic category.

To complete the description of the left spectrum of algebras considered in this section, we need to compute $Spec_l B[x, x^{-1}; \theta_2]$. In order to describe the left spectrum of the skew polynomials considered in this section we need to know the length of the orbit under the action of $\{\theta_2^n\}$ on the left ideals in the list of Lemmas 5.0.2 and 5.0.3 (for the equivalence relation on the left spectrum of a ring). This is only possible in the following cases:

PROPOSITION 5.0.2. *$U_2(k, \zeta)$: Let* $\mathbf{p}$ *be a nonzero left ideal from* (2), (3), *or* (4) *of Lemma* 5.0.2.

(a1) *Let γ be not a root of unity. If* $\mathbf{p}$ *is from either* (2), (3), *or* (4) *of Lemma* 5.0.2, *then $\theta_2^n(\mathbf{p})$ is not equivalent to* $\mathbf{p}$ *for any n.*

(a2) *Let γ be Nth-root of unity. If* $\mathbf{p}$ *is from* (4) *of Lemma* 5.0.2, *then $\theta_2^N(\mathbf{p}) \approx \mathbf{p}$ and N is the smallest integer for which this happens. In other cases, $\theta_2^n(\mathbf{p})$ is not equivalent to* $\mathbf{p}$ *for any n.*

Proof. From the results in [R1, Chap. 2], the left ideals in each of (2), (3), and (4) are non-equivalent to each other. The action of θ_2^n keeps each of these classes stable. Now the above result follows by recalling the action of θ_2: $\theta_2(z) = z - 1$ and $\theta_2(y) = \gamma y$. Since $\theta_2^n(z - a) = z - n - a$, the left ideals in (2) and (3) have an infinite orbit. For a left ideal from (4), $\theta_2^n(y - b) = \gamma^n y - b$. Thus, $\theta_2^N(\mathbf{p}) \approx \mathbf{p}$ iff $\gamma^N = 1$. ∎

PROPOSITION 5.0.3. *The quantum plane: Let* $\mathbf{p}$ *be a non-zero left ideal from* (2), (3), *or* (4) *of Lemma* 5.0.3.

(b1) *Let* $\mathbf{p}$ *be from* (2) *of Lemma* 5.0.3. *Then $\theta_2^j(\mathbf{p}) \approx \mathbf{p}$ for some j iff $\alpha^j = \epsilon^k$ for some k.*

(b2) *Let* $\mathbf{p}$ *be either from* (3) *or* (4) *of Lemma* 5.0.3. *Then $\theta_2^n(\mathbf{p})$ is not equivalent to* $\mathbf{p}$ *for any n iff α and γ are not roots of unity. If either α is Nth-root of unity and* $\mathbf{p}$ *is from* (4) *or ϵ is Nth-root of unity and* $\mathbf{p}$ *is from* (3), *then $\theta_2^N(\mathbf{p}) \approx \mathbf{p}$.*

Proof. Recall that the list of ideals from the left spectrum was obtained by regarding the quantum plane as a hyperbolic ring. The left ideals in (2) correspond to the case of infinite orbit under the action of θ_1. The left ideals in (3) and (4) lie over a θ_1-invariant ideal in $k[w]$. The action of θ_2 keeps these classes stable.

(b1) Let $\mathbf{p}$ be as stated. Recall that $\theta_2(w) = w/\alpha$. By Theorem 2.0.2, $\theta_2^j(\mathbf{p}) \approx \mathbf{p}$ for some j iff $\theta_2^j(w - a) = \theta_1^k(w - a)$. This happens iff $\alpha^j = \epsilon^{-k}$.

(b2) Let $\mathbf{p}$ be from (3) (resp. (4)). By the discussion on left ideals lying over θ-invariant ideals, any two left ideals from either (3) or (4) are non-equivalent [R1]. Then $\theta_2^n(\mathbf{p}) \approx \mathbf{p}$ for some n iff $(\theta_2^n(y - b)) = (y - b)$

$\in k[y]$ (resp. iff $(\theta_2^n(w-a)) = (w-a) \in k[w]$). This happens iff $\gamma^n = 1$ (resp. $\alpha^n = 1$). ∎

To compute $Spec_l B[x, x^{-1}; \theta_2]$, we can treat $B[x, x^{-1}; \theta_2]$ as a hyperbolic ring (over a noncommutative ring) with $\xi = 1$. Thus the description is complete by using the results about the spectrum of hyperbolic categories [R1, Chap. 4]. Observe that for any left ideal $\mathbf{p}$ from $Spec_l B$, $\xi \notin \mathbf{p}$. We have the following proposition.

PROPOSITION 5.0.4. *Let* $\mathbf{p}$ *be any object from* $Spec_l B$. *If* $\theta_2^n(\mathbf{p})$ *is not equivalent to* $\mathbf{p}$ *for any* n, *then* $\theta_2^{\cdot}(\mathbf{p}) = (g, \bigoplus_{i \in \mathbf{Z}} \theta_2^i(\mathbf{p}), h) \in Spec_l B[x, x^{-1}; \theta_2]$ (*cf*. [R1, *Chap*. 4]).

Propositions 5.0.1 and 5.0.2 show that for generic values of parameters, the left ideals from (2), (3), and (4) in Lemma 5.0.2 satisfy the hypothesis of this proposition. For results about the left spectrum lying over $\mathbf{p}$ such that $\theta_2^N(\mathbf{p}) \approx \mathbf{p}$ for some $N \geq 1$, refer to [R1, Sect. 6.7, Theorem 6.9.1, Chap. 4].

The only case we haven't considered here is when ϵ is a root of unity. In that case it is easy to see that there are no "infinite orbits" under the action of θ_2. However, there are left ideals in $\mathbf{p}$ in $Spec_l B$ such that $\theta_2^N(\mathbf{p}) \approx \mathbf{p}$. We refer to [R1, Sect. 6.7, Theorem 6.9.1, Chap. 4] for these cases.

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REFERENCES

[ATV1] M. Artin, J. Tate, and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, *in* "The Grothendieck Festschrift," Vol. 1, pp. 33–85, Birkhäuser, Boston, 1990.

[ATV2] M. Artin, J. Tate, and M. Van den Bergh, Modules over regular algebras of dimension 3, *Invent. Math.* **106** (1991), 335–388.

[L] L. Le Bruyn, Conformal sl_2 enveloping algebras, *Comm. Algebra* **23**, No. 4 (1995), 1325–1362.

[R1] A. L. Rosenberg, "Noncommutative Algebraic Geometry and Representations of Quantized Algebras," Kluwer Academic, Dordrecht, 1995.

[R2] A. L. Rosenberg, Noncommutative local algebra and representations of skew polynomial and hyperbolic rings, preprint.

[W1] E. Witten, Gauge theories and integrable lattice models, *Nuclear Phys. B* **322** (1989), 629–697.

[W2] E. Witten, Gauge theories, vertex models and quantum groups, *Nuclear Phys. B* **330** (1990), 285–346.